NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3557

A THEORETICAL ANALYSIS OF THE FIELD OF A RANDOM NOISE SOURCE ABOVE AN INFINITE PLANE

By Peter A. Franken

Massachusetts Institute of Technology

Washington

November 1955

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3557

A THEORETICAL ANALYSIS OF THE FIELD OF A RANDOM NOISE SOURCE ABOVE AN INFINITE PLANE

By Peter A. Franken

SUMMARY

A theoretical study of the sound field from a random noise source above ground as measured by a receiver with finite band width is presented herein. This study represents one phase of a general program of research in atmospheric acoustics. For simplicity, only the far field has been considered. The special case of a perfectly reflecting plane is discussed first, and nondimensional curves are given of sound pressure level versus distance for two different receiver band widths. The analysis is then extended to the case of a plane of arbitrary impedance, and curves of pressure level versus distance are given for typical field operating conditions. The sound field consists of two major regions. In the first the sound pressure level fluctuates about an average curve sloping approximately 6 decibels per doubling of distance. Beyond a certain distance from the source the level decreases monotonically 12 decibels per doubling of distance. The fluctuations depend on the band width of the receiver and on the ground impedance. With, for example, an octave band of 1,000 to 2,000 cycles and the receiver 10 feet above a ground of normal impedance ρc, the maximum pressure-level fluctuation is about 2 decibels and occurs around 300 feet from the source, and the transition between the 6-decibel-slope region and the 12-decibel-slope region occurs around 700 feet from the source.

INTRODUCTION

The sound field from a point source emitting a pure tone above a plane boundary, which has been studied by several investigators (e.g., ref. 1), can be divided essentially into two parts, one in which there is a marked space variation of sound pressure due to interference between the direct and the reflected sound and one, beyond a certain distance from the source, in which the sound pressure decreases monotonically with distance. In studies of sound propagation in the outdoor atmosphere there is generally more interest in the average rate of decay of the sound pressure with distance from the source and less in level fluctuations in space due to interference. In fact, these fluctuations will generally complicate the interpretation of the data. The sound field can be smoothened by the use of a random noise source rather than a pure tone. However, if the receiver has a finite band width there will still remain fluctuations in space due to interference, a fact which does not seem to have been generally recognized. It is the purpose of this paper to analyze this problem theoretically and, in particular, to determine the level fluctuations as a function of receiver band width for the case of a random noise source with a constant power spectrum.

The present investigation was conducted at the Acoustics Laboratory of the Massachusetts Institute of Technology under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

SYMBOLS

$$C = \frac{\pi}{c} \frac{\left(\cos \theta_o + \beta\right)^2}{1 + \beta \cos \theta_o}\left(d + \frac{\Delta}{2}\right)$$

c velocity of sound above plane

d mean source-to-receiver distance

f frequency

f_m mean frequency of band pass filter

H height of receiver above plane

h height of source above plane

$\overline{P^2}$ resultant mean square pressure

p sound pressure

Q image source strength

R ratio of mean square pressure with plane present to free space mean square pressure

R_o plane wave reflection coefficient

r ratio of band-pass-filter end points

S	pressure spectrum
T	upper limit of time in pressure-spectrum definition
t	time
w	power spectrum
x	horizontal source-to-receiver distance
β	admittance ratio of reflecting plane
Δ	path-length difference
θ_o	grazing angle of reflected ray
λ	wavelength of sound
λ_m	wavelength corresponding to f_m
ρc	characteristic impedance of air
τ	delay time
ψ	correlation function

ANALYSIS

The sound field about a point source above an infinite plane can be obtained as the sum of the direct wave and the wave reflected from an image source below the plane (ref. 1). With the geometry defined in figure 1, the expression for the path difference between the direct and the reflected wave is

$$\Delta = 2hH/d \tag{1}$$

Limiting the discussion to far-field points imposes the restriction $\Delta \ll d$ or

$$2hH/d^2 \ll 1 \tag{2}$$

In the discussion which follows it is also assumed that h and H are of the same order of magnitude so that $d \approx x$; however, the analysis is not necessarily restricted to these conditions.

Suppose that the source emits a random noise and that the corresponding sound pressure of the direct wave at the point of observation is the random function of time $p(t)$. The pressure spectrum $S(f)$ of $p(t)$ is defined (ref. 2) as

$$S(f) = \int_0^T p(t)e^{-2\pi ift}\,dt \tag{3}$$

and the power spectrum $w(f)$ is then

$$w(f) = \lim_{T \to \infty} \frac{2|S(f)|^2}{T} \tag{4}$$

Since $p(t)$ is assumed to be perfectly random and to contain no periodic terms, this definition of $w(f)$ is adequate. The correlation function $\psi(\tau)$ of $p(t)$ is defined as

$$\psi(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T p(t)p(t + \tau)\,dt \tag{5}$$

where $p(t + \tau)$ is the value of the pressure at a time τ later at the same point. The power spectrum and correlation function are related by Fourier formulas

$$w(f) = 4 \int_0^\infty \psi(\tau) \cos 2\pi f\tau \,d\tau \tag{6}$$

$$\psi(\tau) = \int_0^\infty w(f) \cos 2\pi f\tau \,df \tag{7}$$

The case of a perfectly reflecting plane is considered first, since it illustrates the general approach to the problem while simplifying the mathematics involved. The discussion will then be generalized to the case of a plane of arbitrary impedance.

Perfect Reflector

With a perfectly reflecting plane present, it is seen qualitatively that $\psi(\tau)$ is a measure of the interference between the direct ray at any point and the reflected ray which starts out τ seconds earlier. If the pressure-measuring device had an infinite band width, there would be no correlation between the two rays except for $\tau = 0$, or

$$\psi_p(\tau) = \int_0^\infty w(f)\, df \qquad (\tau = 0) \qquad (8a)$$

$$\psi_p(t) = 0 \qquad (\tau \neq 0) \qquad (8b)$$

Now, consider a receiver with a band pass filter with a range from f_a to f_b. In such a case, using equation (7), the correlation function becomes

$$\begin{aligned}\psi(\tau) &= \int_{f_a}^{f_b} w_o \cos 2\pi f\tau\, df \\ &= \frac{w_o}{2\pi\tau}\left(\sin 2\pi f_b\tau - \sin 2\pi f_a\tau\right) \qquad (9)\end{aligned}$$

where w_o is the constant value of $w(f)$. The mean square pressure at a point $\overline{P^2}$ is

$$\overline{P^2(\tau)} = \overline{\left[p(t) + p(t + \tau)\right]^2}$$

$$= \left[\overline{p(t)^2} + \overline{p(t + \tau)^2} + 2\,\overline{p(t)p(t + \tau)}\right]$$

$$= \left[\overline{p(t)^2} + \overline{p(t + \tau)^2} + 2\psi(\tau)\right] \tag{10}$$

where the bars indicate time averages. Only far-field points are considered, that is, points where $\overline{p(t)^2} \approx \overline{p(t + \tau)^2}$ or the path-length difference between the direct and reflected rays $\Delta(= c\tau)$ is small compared with the source-to-receiver distance for either ray. Also,

$$\overline{p(t)^2} = \psi(0) = w_o\left(f_b - f_a\right) \tag{11}$$

In the case of free space,

$$\overline{P_F{}^2} = \overline{p(t)^2} = w_o\left(f_b - f_a\right) \tag{12}$$

The ratio of mean square pressure when the plane is present to free-space mean square pressure is, combining the results of equations (9) to (12),

$$R(\tau) = \frac{\overline{P^2(\tau)}}{\overline{P_F{}^2}} = 2 + \frac{2}{\left(f_b - f_a\right)2\pi\tau}\left(\sin 2\pi f_b\tau - \sin 2\pi f_a\tau\right) \tag{13}$$

Utilizing a trigonometric identity, equation (13) becomes

$$R = 2 + 2\left[\frac{\sin\left(2\pi f_m\tau\,\frac{r - 1}{r + 1}\right)}{2\pi f_m\tau\,\frac{r - 1}{r + 1}}\cos 2\pi f_m\tau\right] \tag{14}$$

where

$$f_m = \frac{f_a + f_b}{2}$$

$$r = \frac{f_b}{f_a}$$

Two important limiting cases are: (a) $\tau = 0$, in which direct and reflected rays interfere constructively and $R = 4$ and (b) $r = 1$, in which the pass filter becomes infinitesimally narrow or, equivalently, the source becomes monochromatic, with the power spectrum becoming infinite so that $\int w(f)\,df$ remains finite and

$$R = 2 + 2 \cos 2\pi f_m \tau \tag{15}$$

which is the correct expression for a pure-tone source.

Equation (14) expresses R as a function of the two dimensionless variables r and $f_m\tau$. The free-space mean square sound pressure $\overline{P_F{}^2}$ is inversely proportional to x^2. Plots can be made of the recorded sound pressure level $10 \log \overline{P_F{}^2}R$ versus the "numerical distance" $\lambda_m x/hH$, with r as a parameter, since, using equation (2),

$$f_m\tau = \frac{c}{\lambda_m}\frac{\Delta}{c} \approx \frac{2hH}{\lambda_m x} \tag{16}$$

where λ_m is the wavelength corresponding to f_m. Figure 2 shows such plots for an octave-band pass filter ($r = 2$) and a half-octave-band pass filter $\left(r = \frac{3}{2}\right)$. The curves approach asymptotically the $1/x^2$ curve shown as a short-dashed line. The deviations for the narrower filter are more pronounced than those for the broader filter, as expected. The form of the variable $\lambda_m x/hH$ and the general shape of the curves indicate that, for a given distance x and band-width ratio r, decreasing the mean filter frequency f_m or the heights h and H tends to increase the magnitude of the intensity deviations.

Plane of Arbitrary Impedance

The sound field of a pure tone above an infinite plane of normal specific acoustic impedance $\rho c/\beta$ is found by Ingard to be due to the real point source and an image source of strength Q, where Q is a function of the field point position, the plane impedance, and the source frequency. This result is adapted to the noise problem, considering again only far-field points. With the source in the plane, $\Delta = 0$, and the pressure spectrum of the real source is as before $S(f)$ (eq. (3)), while the pressure spectrum of the image source is $S(f)Q(d,\beta,f)$. The inverse of equation (3) permits the transformation from pressure spectra to pressures, and the resultant mean square pressure at a point is

$$\overline{P^2} = \overline{\left[\int_0^\infty S(1 + Q)\, e^{2\pi ift}\, df\right]^2}$$

$$= \lim_{T \to \infty} \frac{2}{T} \int_0^\infty \left|S(1 + Q)\right|^2 df$$

$$= w_o \left\{\int_{f_a}^{f_b} \left[1 + \mathrm{Re}(Q)\right]^2 df + \int_{f_a}^{f_b} \left[\mathrm{Im}(Q)\right]^2 df\right\} \tag{17}$$

where w_o is the constant value of the power spectrum (eq. (4)) in the pass band f_a to f_b and Re and Im signify real and imaginary parts, respectively.

The integrals in equation (17) may be evaluated by use of the asymptotic form of Q, valid at large distances, which is

$$Q = R_o + \left(1 - R_o\right)\left(\frac{-i}{2fC} + \frac{3}{4f^2C^2} + \ldots\right) \tag{18}$$

where

$$C = \frac{\pi}{c} \frac{(\cos \theta_o + \beta)^2}{1 + \beta \cos \theta_o} \left(d + \frac{\Delta}{2}\right)$$

and R_o is the plane wave reflection coefficient, a function of position. Keeping lowest order terms only, equation (17) becomes

$$\overline{P^2} = w_o\left[\left(f_b - f_a\right)\left(1 + R_o\right)^2 + \frac{1}{4}\left(1 - R_o\right)\left(5R_o + 7\right)\left(\frac{f_b - f_a}{C^2 f_a f_b}\right)\right] \tag{19}$$

Dividing equation (19) by equation (12), the measured free-space mean square pressure, yields

$$R = \left(1 + R_o\right)^2 + \frac{\left(1 - R_o\right)\left(5R_o + 7\right)}{4C^2 f_a f_b} \tag{20}$$

The first term in this equation is the ray-acoustics expression for plane wave reflection, and the second term is a small correction factor which reduces monotonically with distance.

With the source above the plane, the pressure spectrum for the image ray is

$$\int_0^T Qp(t + \tau)e^{-2\pi ift}\, dt = Qe^{2\pi if\tau}S \tag{21}$$

The total pressure corresponds to the pressure spectrum $\left(1 + Qe^{2\pi if\tau}\right)S$, and the power spectrum becomes $w_o\left(1 + Qe^{2\pi if\tau}\right)^2$. The mean square pressure, with filter limits f_a and f_b, is then

$$\overline{P^2} = w_o \int_{f_a}^{f_b} \left\{ 1 + 2\mathrm{Re}(Q) \cos 2\pi f\tau - 2\mathrm{Im}(Q) \sin 2\pi f\tau + \left[\mathrm{Re}(Q)\right]^2 + \left[\mathrm{Im}(Q)\right]^2 \right\} df \tag{22}$$

Following the same procedure used to evaluate equation (17), the ratio of mean square pressure to free-space mean square pressure becomes in this case

$$R = 1 + R_o^2 + 2R_o \frac{\psi(\tau)}{w_o(f_b - f_a)} + \frac{(1 - R_o)}{C} 2\pi\tau + \frac{(1 - R_o)(5R_o + 7)}{4C^2 f_a f_b} + \frac{3(1 - R_o)}{2C^2} \left[\frac{\cos 2\pi f_a\tau}{f_a(f_b - f_a)} - \frac{\cos 2\pi f_b\tau}{f_b(f_b - f_a)} + (2\pi\tau)^2 \right] \tag{23}$$

For $\tau = 0$, this expression reduces to equation (20) for the source in the plane. For $R_o = 1$, it reduces to equation (13) for the perfectly reflecting plane.

Numerical Examples

In order to evaluate R in equation (23) an octave-band pass filter is chosen with $f_a = 1{,}000$ cps, $f_b = 2{,}000$ cps, and source and receiver heights $h = H = 10$ feet. In this case the terms in equation (23) containing a frequency squared in the denominator or τ in the numerator are negligible in the far field, or

$$R \approx 1 + R_o^2 + 2R_o \frac{\psi(\tau)}{w_o(f_b - f_a)} \tag{24}$$

(This is equivalent to setting $Q = R_o$.) For a pure tone of frequency f_m

$$R = 1 + R_o^2 + 2R_o \cos 2\pi f_m \tau \tag{25}$$

by the same limiting process involved in equations (14) and (15).

In figures 3(a) to 3(d) plots of the measured sound pressure level $10 \log \overline{P_F^2} R$ versus source-to-receiver distance x are given for the perfect reflector ($\beta = 0$) and for the admittance ratios $\beta = 1/4$, $1/2$, and 1. The broken lines correspond to random noise and the solid line corresponds to a pure tone of the mean frequency of the band pass filter.

Figures 3(e) and 3(f) are similar plots for $\beta = 1$, $h = H = 10$ feet, and octave bands 75 to 150 cps and 300 to 600 cps. For these bands the terms in equation (23) containing τ in the numerator are small but not always negligible.

It can be seen from figures 3(b) to 3(f) that above a plane of finite impedance the sound pressure level measured near the source fluctuates about an average curve sloping approximately 6 decibels per doubling of distance and that at large distances from the source the measured sound pressure level approaches a decrease of 12 decibels per doubling of distance. In particular, figure 3(d) indicates that for an octave band of 1,000 to 2,000 cycles, with the source and receiver 10 feet above a ground of normal impedance ρc, the maximum pressure-level fluctuation is about 6 decibels and occurs around 300 feet from the source and that the transition between the 6-decibel-slope region and the 12-decibel-slope region occurs around 700 feet from the source. The 12-decibel slope at large distances can be predicted from the following expressions for the image source Q and for the resultant pressure P in the far field:

$$Q \approx R_o = \frac{\cos \theta_o/(\beta - 1)}{\cos \theta_o/(\beta + 1)} \approx -\left(\frac{2H}{\beta x} - 1\right)^2 \approx -1 + \frac{4H}{\beta x} \tag{26}$$

$$|P| \approx \frac{1}{x}(1 + Q) \approx \frac{1}{x^2}\frac{4H}{\beta} \tag{27}$$

Massachusetts Institute of Technology,
Cambridge, Mass., January 11, 1955.

REFERENCES

1. Ingard, Uno: On the Reflection of a Spherical Sound Wave From an Infinite Plane. Jour. Acous. Soc. Am., vol. 23, no. 3, May 1951, pp. 329-335.

2. Rice, S. O.: Mathematical Analysis of Random Noise. The Bell System Tech. Jour., vol. 23, no. 3, July 1944, pp. 282-332.

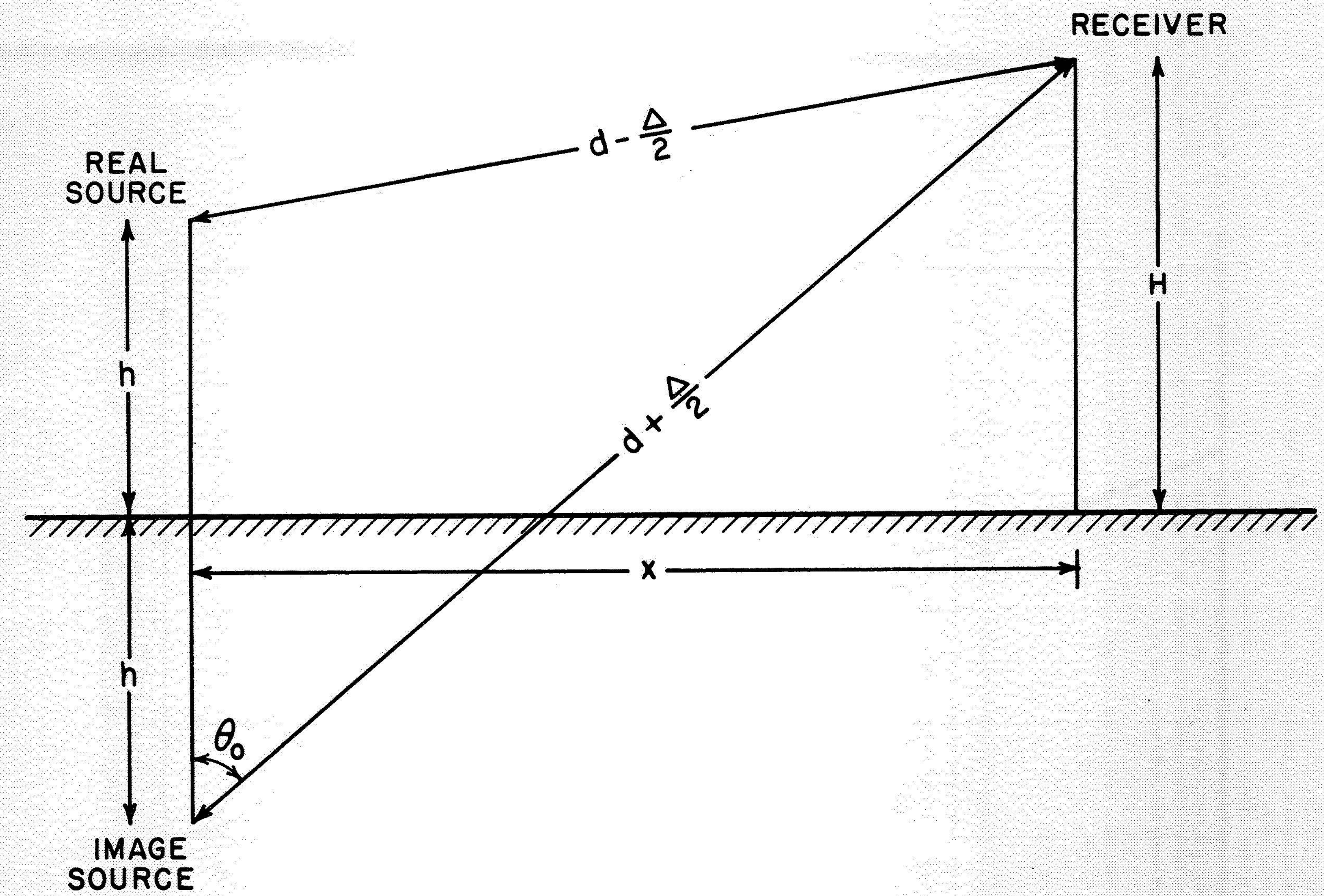


Figure 1.- Geometry of point source above infinite plane showing direct and image rays.

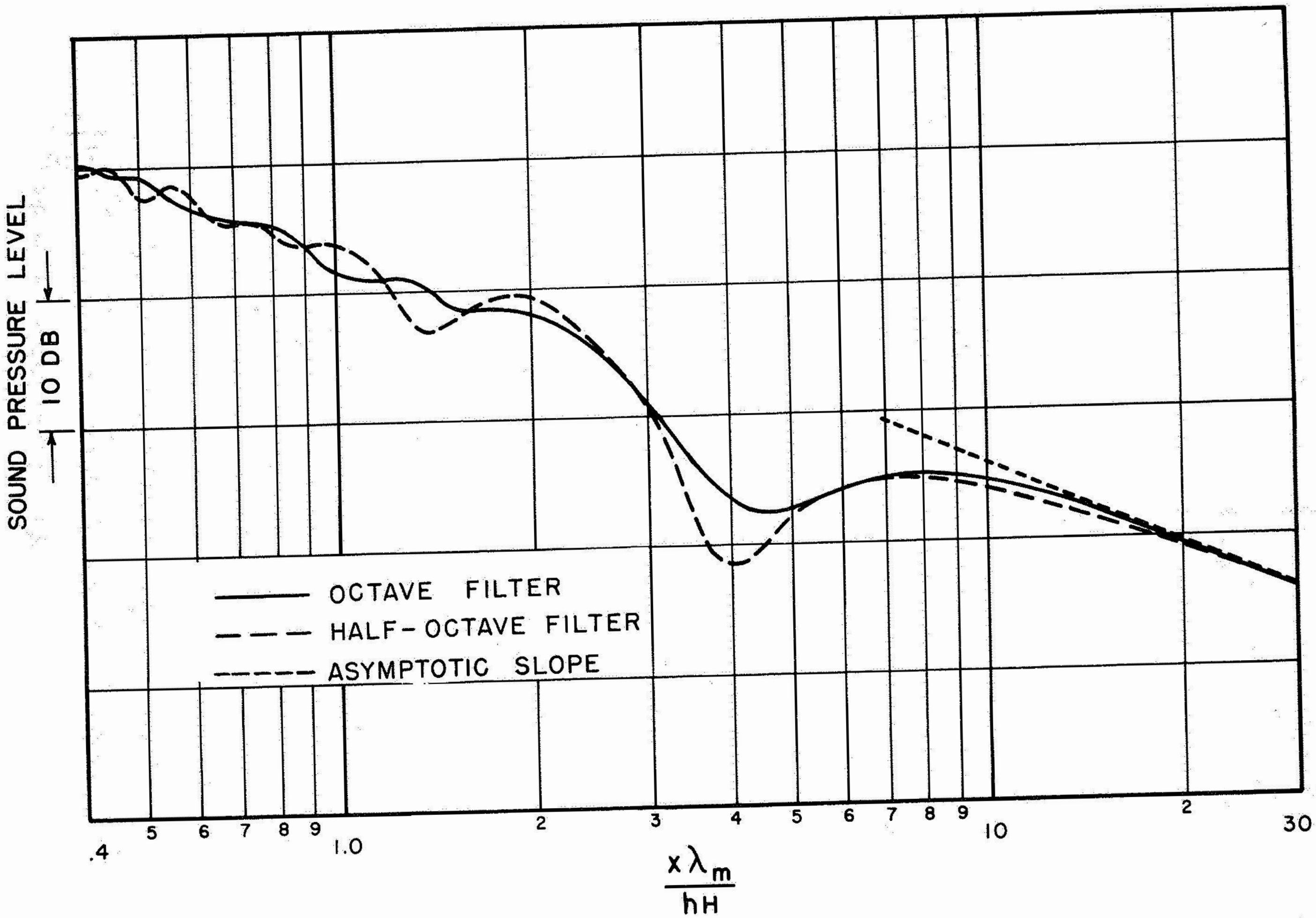
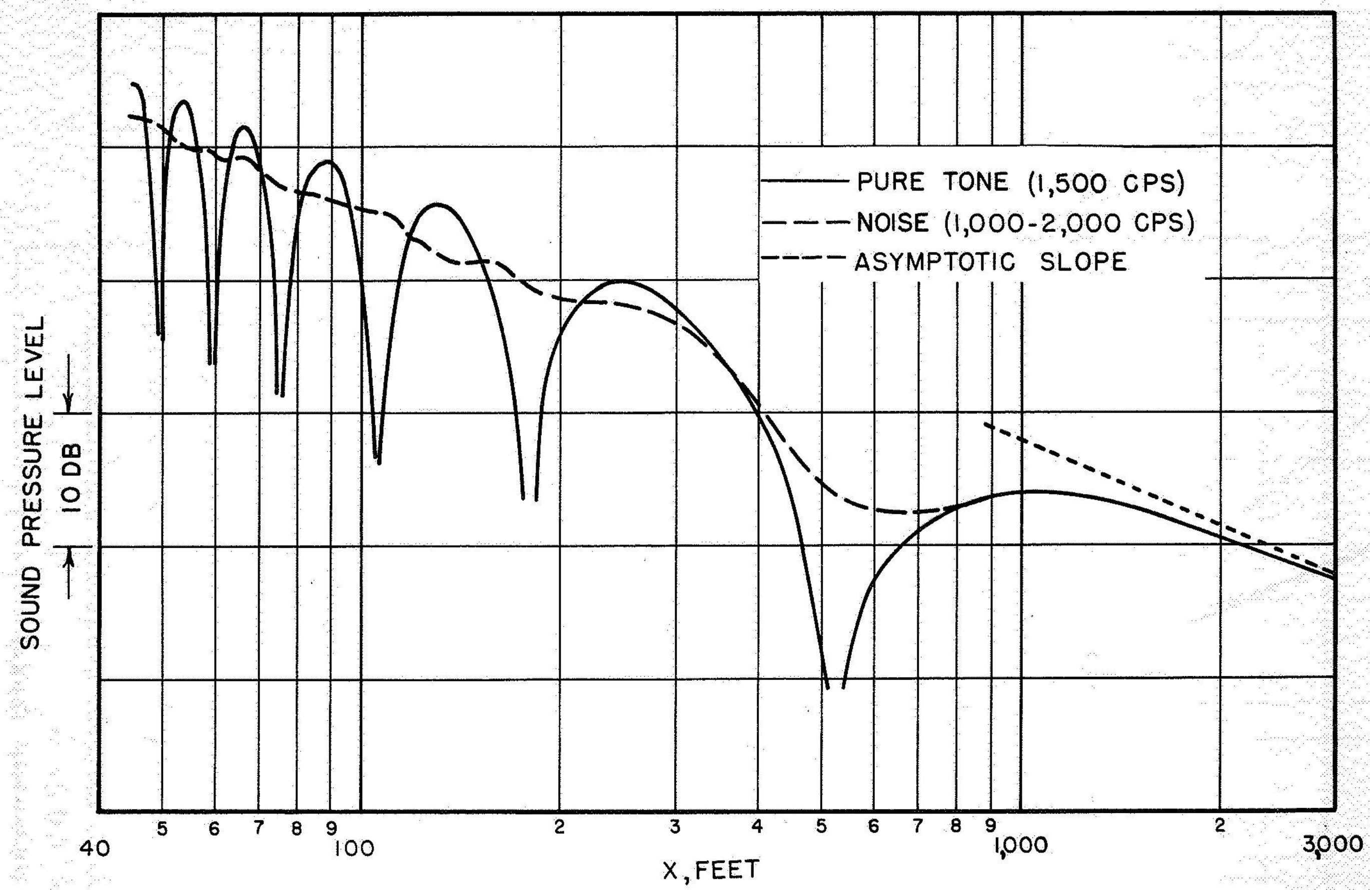
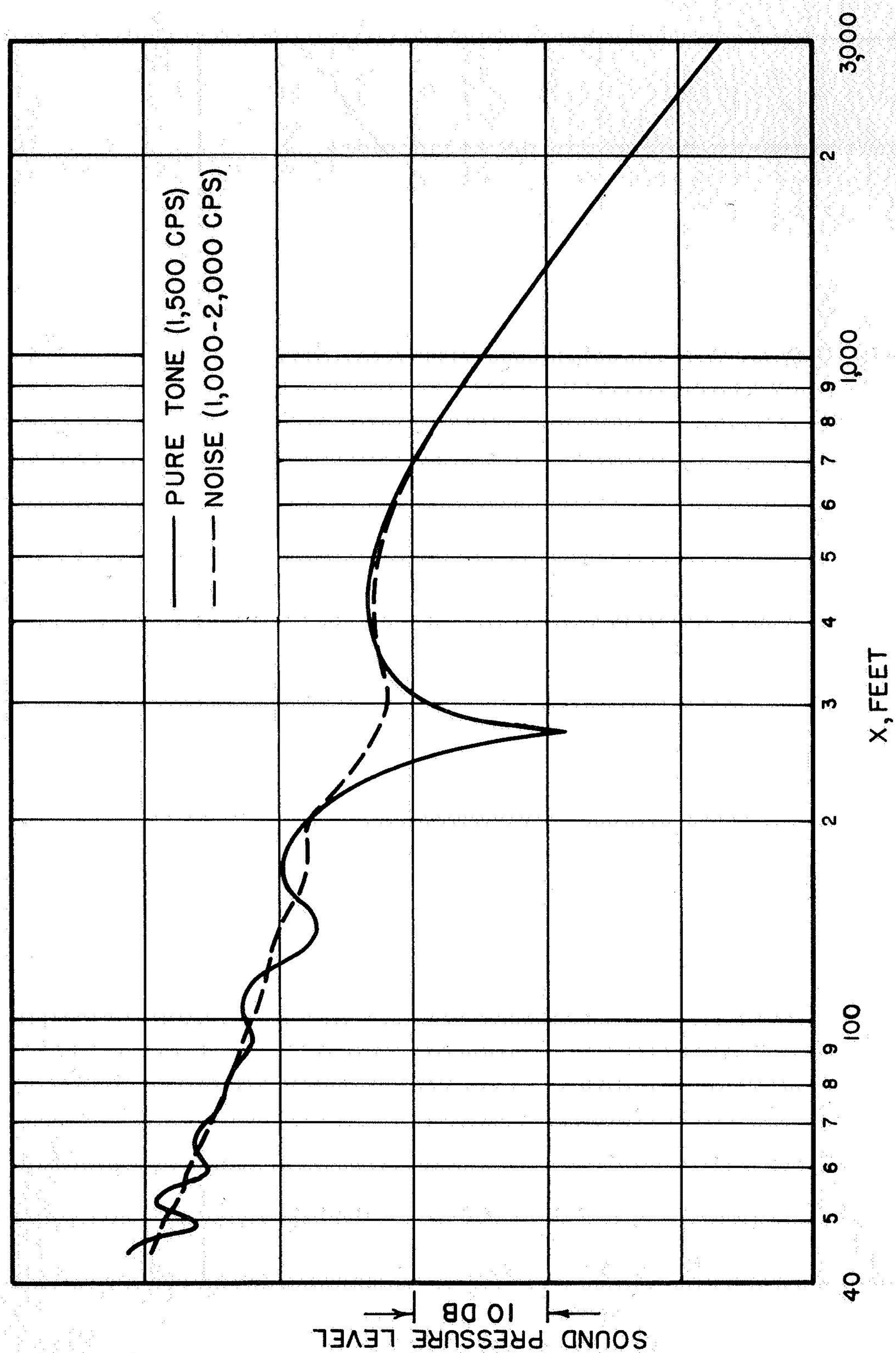


Figure 2.- Sound pressure level $10 \log \overline{P_F^2}R$ plotted versus the "numerical distance" $\lambda_m x/hH$ in the case of a perfectly reflecting plane. Solid line indicates octave-band pass filter ($r = 2$); broken line indicates half-octave filter ($r = 3/2$); $\beta = 0$.



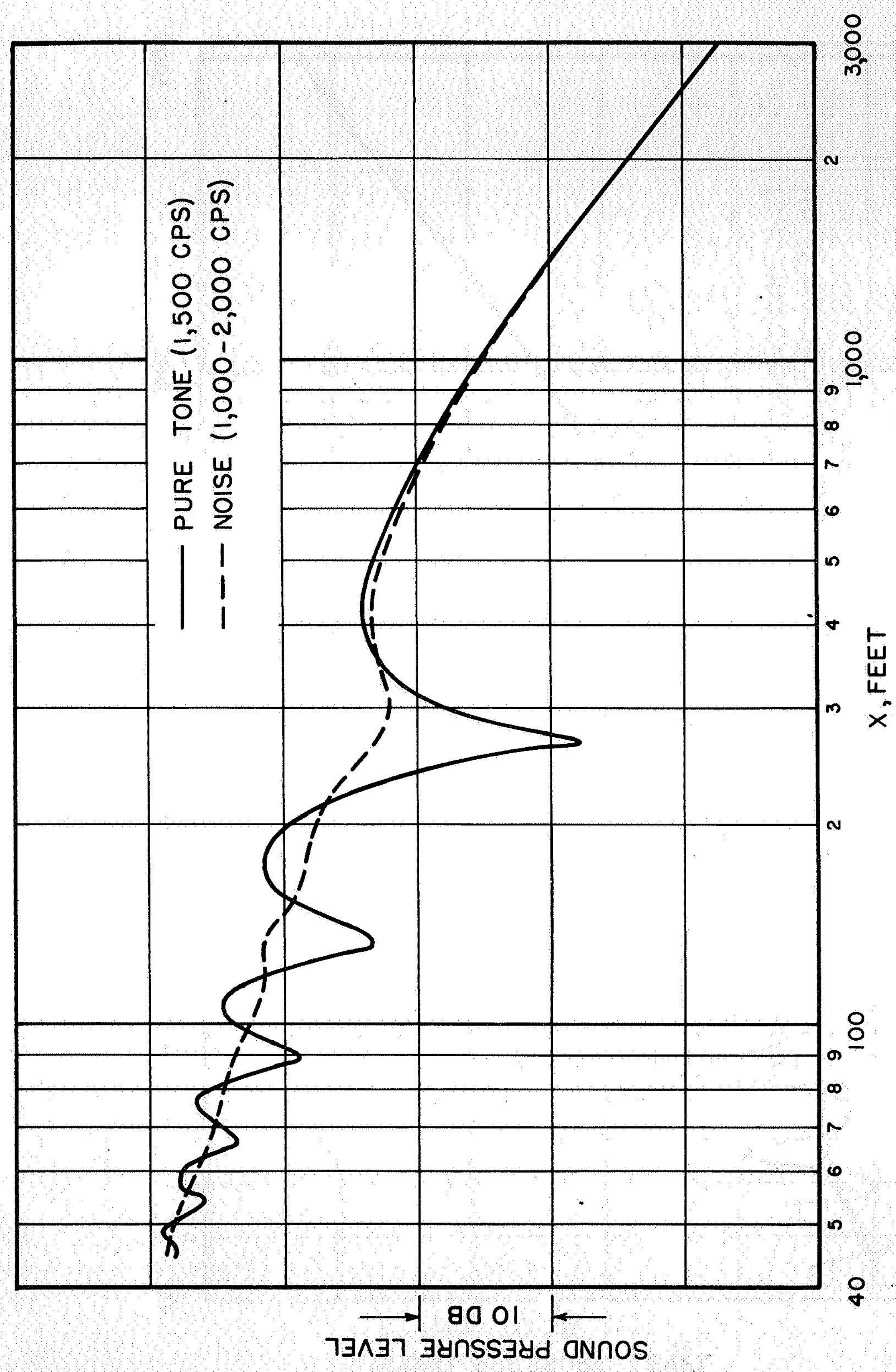
(a) Octave band, 1,000 to 2,000 cps; $\beta = 0$.

Figure 3.- Sound pressure level $10 \log \overline{P_F^2}R$ plotted versus source-to-receiver distance x for source and receiver heights of 10 feet. Broken line corresponds to random noise; solid line indicates pure tone of mean filter frequency.



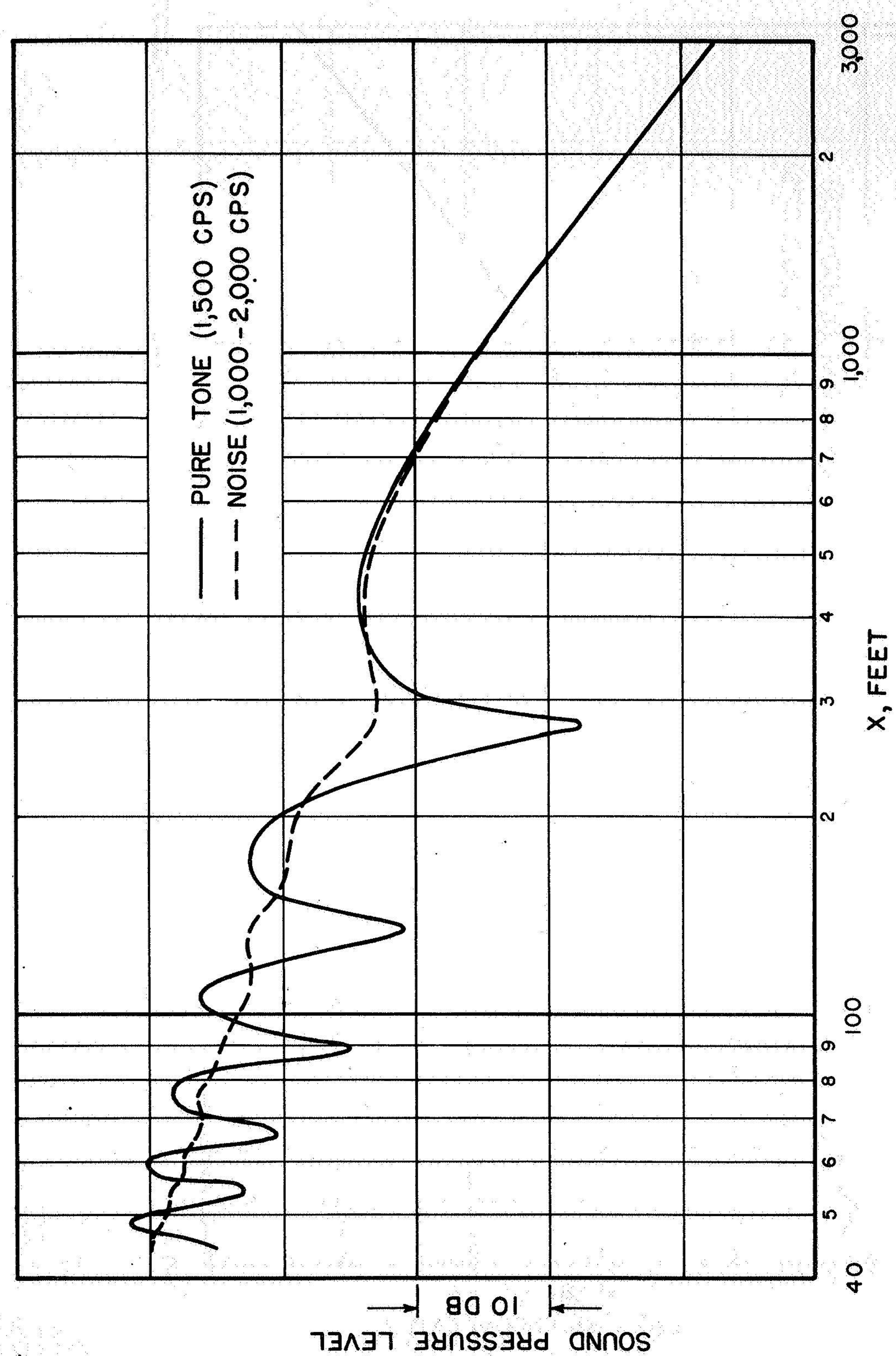
(b) Octave band, 1,000 to 2,000; β = 1/4.

Figure 3.- Continued.



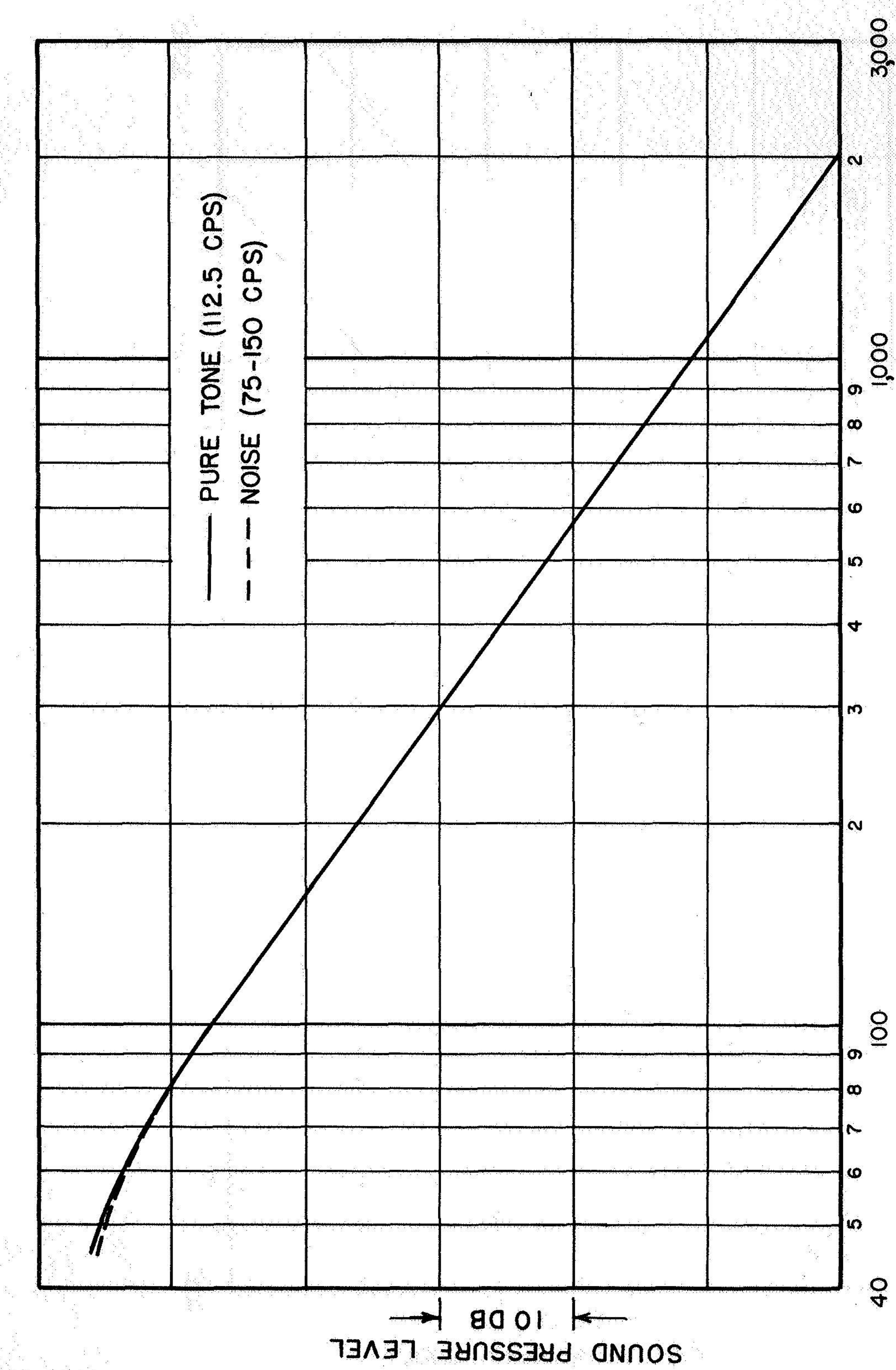
(c) Octave band, 1,000 to 2,000; β = 1/2.

Figure 3.- Continued.



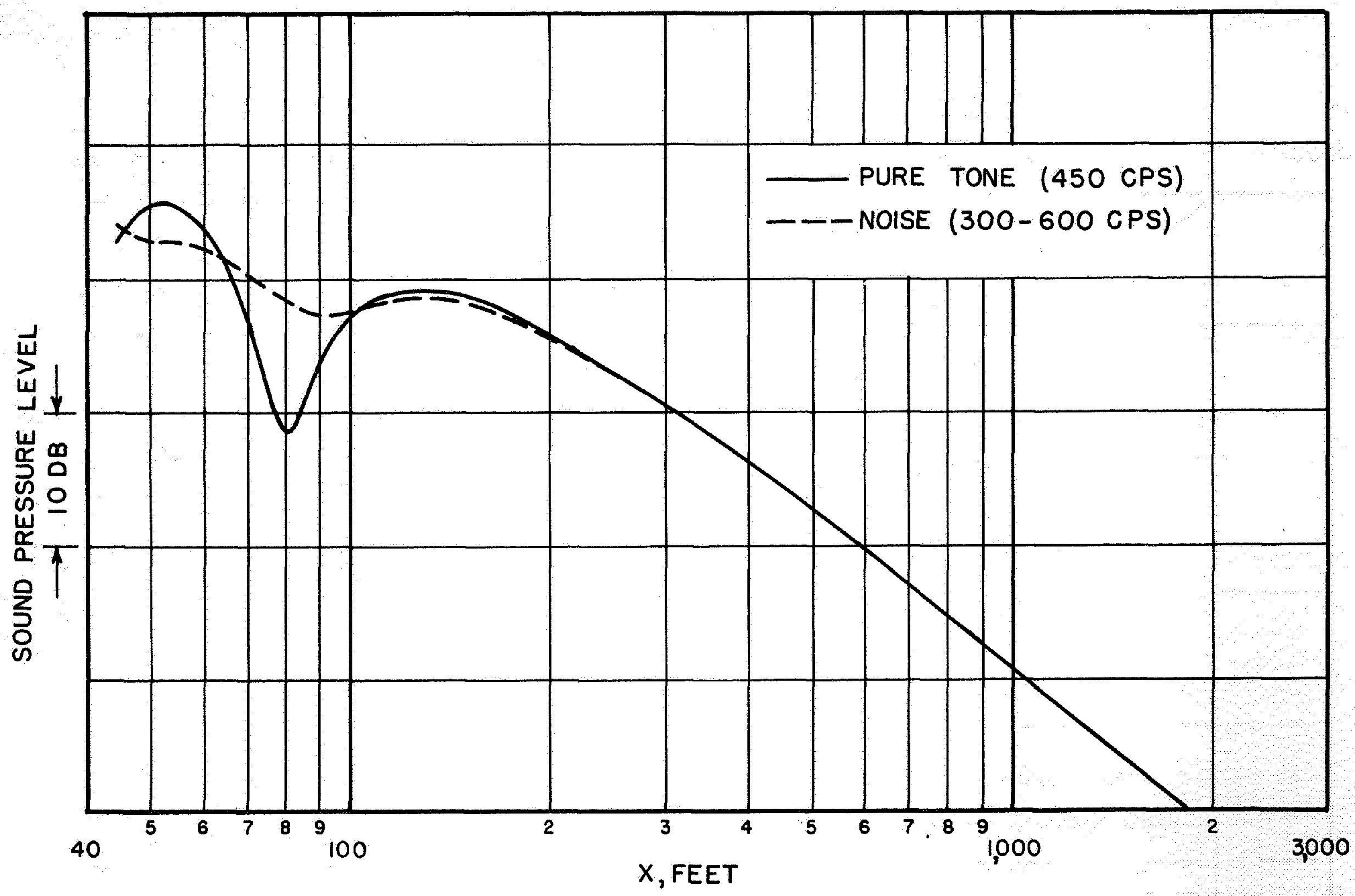
(d) Octave band, 1,000 to 2,000; β = 1.

Figure 3.- Continued.



(e) Octave band, 75 to 150 cps; β = 1.

Figure 3.- Continued.



(f) Octave band, 300 to 600 cps; $\beta = 1$.

Figure 3.- Concluded.